

FREE CONVEX SETS DEFINED BY RATIONAL EXPRESSIONS HAVE LMI REPRESENTATIONS

J. WILLIAM HELTON¹ AND SCOTT MCCULLOUGH²

ABSTRACT. Suppose p is a symmetric matrix whose entries are polynomials in freely non-commutative variables and $p(0)$ is positive definite. Let \mathcal{D}_p denote the component of zero of the set of those g -tuples $X = (X_1, \dots, X_g)$ of symmetric matrices (of the same size) such that $p(X)$ is positive definite. In [HM12] it was shown that if \mathcal{D}_p is convex and bounded, then \mathcal{D}_p can be described as the set of solutions of a linear matrix inequality (LMI). This article extends that result from matrices of polynomials to matrices of rational functions in free variables.

As a refinement of a theorem of Kaliuzhnyi-Verbovetskyi and Vinnikov, it is also shown that a minimal symmetric descriptor realization r for a symmetric free matrix-valued rational function \mathfrak{r} in g freely noncommuting variables $x = (x_1, \dots, x_g)$ precisely encodes the singularities of the rational function. This singularities result is an important ingredient in the proof of the LMI representation theorem stated above.

1. INTRODUCTION

Given positive integers g and n , let $x = (x_1, \dots, x_g)$ denote g freely noncommuting indeterminates and let $\mathbb{S}_n(\mathbb{R}^g)$ denote the g -tuples $X = (X_1, \dots, X_g)$ of symmetric $n \times n$ matrices (with real entries). Given a positive integer d and a tuple $A \in \mathbb{S}_d(\mathbb{R}^g)$, let $L_A(x)$ denote the **homogeneous linear pencil**

$$L_A(x) = \sum_{j=1}^g A_j x_j.$$

The $d \times d$ matrix J is a **symmetry** if $J = J^T$ and $J^2 = I$. The expression

$$J - L_A(x)$$

Date: November 22, 2012.

2010 *Mathematics Subject Classification.* 47Axx (Primary) 47A63, 47L07, 14P10 (Secondary).

Key words and phrases. matrix convexity, free convexity, linear matrix inequality, noncommutative rational function, free rational function.

¹Research supported by NSF grants DMS-0700758, DMS-0757212, and the Ford Motor Co.

²Research supported by NSF grant DMS-1101137.

is an example of an **affine linear pencil**. Such a pencil is naturally evaluated at $X \in \mathbb{S}_n(\mathbb{R}^g)$ as

$$J - L_A(X) = J \otimes I_n - \sum_{j=1}^g A_j \otimes X_j,$$

with the result a $dn \times dn$ symmetric matrix. Here \otimes denotes the Kronecker product of matrices and I_n the $n \times n$ identity matrix.

For a symmetric matrix Y , the notation $Y \succ 0$ (resp. $Y \succeq 0$) indicates that Y is positive definite (resp. positive semidefinite). In the case that $J = I$, the pencil $I - L_A$ is a **monic affine linear pencil** and, for each n , the set

$$\mathcal{D}_{I-L_A}(n) = \{X \in \mathbb{S}_n(\mathbb{R}^g) : I - L_A(X) \succ 0\}$$

is convex. The sequence of sets $(\mathcal{D}_{I-L_A}(n))_n$ is a **(free) LMI domain**, synonymously a **(free) spectrahedron**.

The main results of this article gives an LMI representation for a free bounded convex set determined by a free rational function. Before giving a precise statement, we pause to informally describe a sample result. Let \mathfrak{r} be an $\ell \times \ell$ matrix-valued symmetric free rational function which is positive definite at the origin. *If the sequence of sets*

$$\mathfrak{P}_{\mathfrak{r}}(n) = \{X \in \mathbb{S}_n(\mathbb{R}^g) : \mathfrak{r}(X) \text{ is defined and } \mathfrak{r}(X) \succ 0\}$$

is uniformly bounded and each $\mathfrak{P}_{\mathfrak{r}}(n)$ is convex, then there is a monic affine linear pencil $I - L_A$ such that, for each n ,

$$\mathfrak{P}_{\mathfrak{r}}(n) = \mathcal{D}_{I-L_A(x)}(n).$$

The proof depends upon a description of the domain and singularities of a free rational function together with the main result of [HM12]. In the remainder of this introduction the fairly minimal necessary background on free rational functions - in terms of descriptor realizations - and their singularities needed to state the main results is developed. Descriptor realizations and their domains are the topic of Subsections 1.1 and 1.2 respectively. A precise statement of the main results on LMI representations of free bounded convex sets determined by free rational functions is given in Subsection 1.3 as Theorem 1.2. The proof of Theorem 1.2 hinges upon the nature of the singularities of descriptor realizations a discussion of which appears in Subsection 1.4. Subsection 1.5 is a reader's guide to the body of the paper. Acknowledgments appear in Section 1.6.

1.1. Descriptor realizations. For a positive integer n , the set

$$(1.1) \quad \mathfrak{J}_{J-L_A(x)}(n) = \{X \in \mathbb{S}_n(\mathbb{R}^g) : J - L_A(X) \text{ is invertible}\}$$

is open and dense in $\mathbb{S}_n(\mathbb{R}^g)$. The sequence $\mathfrak{J}_{J-L_A(x)} = (\mathfrak{J}_{J-L_A(x)}(n))_n$ is the **invertibility set** of $J - L_A(x)$. Let $M_{d \times \ell}$ denote the $d \times \ell$ matrices with real entries. Given $C \in M_{d \times \ell}$ and $D \in M_{\ell \times \ell}$ the expression

$$(1.2) \quad r(x) = D + C^T(J - L_A(x))^{-1}C$$

is known as a **descriptor realization**.^{1 2} It is an example of a **rational expression**. We assume throughout that that r is **positive at the origin**; i.e.,

$$D + C^T J C \succ 0.$$

The rational expression r can be evaluated at any $X \in \mathfrak{J}_{J-L_A(x)}$ as

$$r(X) = D \otimes I + C^T \otimes I(J - L_A(X))^{-1}C \otimes I.$$

In the case that X has size n so that $X \in \mathfrak{J}_{J-L_A(x)}(n)$, the matrix I is the $n \times n$ identity and the matrix $r(X)$ is symmetric of size $\ell n \times \ell n$. Further, for $X = \mathbf{0}$ - the g -tuple of $n \times n$ zero matrices - and $0 \in \mathbb{R}^g$ the 0 vector,

$$r(\mathbf{0}) = r(0) \otimes I_n = (D + C^T J C) \otimes I_n \succ 0.$$

1.2. The domains of a descriptor realization. It could happen that there is an n and $\chi \notin \mathfrak{J}_{J-L_A(x)}(n)$ such that

$$r(\chi) := \lim_{X \rightarrow \chi} r(X)$$

exists, where the limit is taken through $\mathfrak{J}_{J-L_A(x)}(n)$. Such a χ is called a **hidden singularity** of the rational expression r . For positive integers n , let

$$\text{Domlim}(r, n) = (\mathfrak{J}_{J-L_A(x)}(n) \cup \text{hidden singularities of } r) \cap \mathbb{S}_n(\mathbb{R}^g).$$

The **limit domain** of r , denoted $\text{Domlim}(r, n)$, is the sequence $(\text{Domlim}(r, n))_n$.

The notion of a rational expression (analytic at 0)³ is defined in Section 6. For now we note that a descriptor realization is an example and that rational expressions have, for each n , a domain in $\mathbb{S}_n(\mathbb{R}^g)$ which is both open and dense and contains a neighborhood of 0. Two rational expressions r and \hat{r} may have the property $r(X) = \hat{r}(X)$ for all tuples of matrices for which they are both defined, in which case we shall say they represent the same noncommutative (free) rational function \mathfrak{r} . Thus, a free rational function \mathfrak{r} is an equivalence class of rational expressions and it is natural to define another notion of domain for the rational expression r . The **algebraic domain** of r is the union of the formal domains of all the rational expressions equivalent to r . A more precise presentation, complete with the

¹More precisely r is a **symmetric** descriptor realization, but we often drop the adjective symmetric here.

²The reason for this terminology is explained later.

³In this article, rational expression means rational expression analytic at 0.

definition of formal domain, appears in Section 6. In any event, the algebraic domain of the descriptor realization r is also a sequence $\text{Domalg}(r) = (\text{Domalg}(r, n))_n$ and

$$\text{Domalg}(r, n) \subseteq \text{Domlim}(r, n) \subseteq \mathbb{S}_n(\mathbb{R}^g).$$

Remark 1.1. For a classical perspective on the distinction between limit and algebraic domains, consider the rational function

$$R(x, y) = \frac{x^2 y^2}{x^2 + y^2}.$$

The origin is in its limit domain, but not in its algebraic domain.

Generally, suppose P and Q are classical commutative relatively prime polynomials in g variables. A *non-essential singularity of the second kind* of the rational function $R = \frac{P}{Q}$ is a point where both P and Q vanish. There is a large literature in multivariable systems theory of (classical commutative) rational functions (transfer functions) with essential singularities of the second kind at points on the distinguished boundary of the polydisc.

1.3. LMI representations. For positive integers n , let

$$\mathfrak{P}_r(n) = \{X \in \text{Domlim}(r, n) : r(X) \succ 0\}.$$

The sequence $\mathfrak{P}_r = (\mathfrak{P}_r(n))$ is the **positivity set** of r .

Similarly, let $\mathcal{D}_r(n)$ denote the component of zero of the set

$$\{X \in \text{Domalg}(r, n) : r(X) \succ 0\};$$

that is, $\mathcal{D}_r(n)$ is the principal component of the the set of $X \in \mathbb{S}_n(\mathbb{R}^g)$ in the algebraic domain of r for which $r(X)$ is positive definite. The assumption that $r(0) \succ 0$ implies the sets $\mathcal{D}_r(n)$ and $\mathfrak{P}_r(n)$ are not empty (for each n).

A sequence $\mathcal{D} = (\mathcal{D}(n))$ of sets $\mathcal{D}(n) \subseteq \mathbb{S}_n(\mathbb{R}^g)$ is **bounded** if there exists an $R \in \mathbb{R}$ such that $X_1^2 + \cdots + X_g^2 \preceq RI_n$ for each n and $X \in \mathcal{D}(n)$. The following theorem contains the main results of this article. The notion of a minimal descriptor realization is defined at the outset of Section 2. For now we note that the condition is natural and that minimality can be assumed without loss of generality.

Theorem 1.2. *Suppose r is a minimal symmetric descriptor realization with $r(0) \succ 0$.*

(lim) *If \mathfrak{P}_r is bounded and each $\mathfrak{P}_r(n)$ is convex, then there exists a positive integer m and a tuple $\mathbb{A} \in \mathbb{S}_m(\mathbb{R}^g)$ such that $X \in \mathfrak{P}_r(n)$ if and only if*

$$I - L_{\mathbb{A}}(X) \succ 0.$$

(alg) If \mathcal{D}_r is bounded and for each positive integer n the set $\mathcal{D}_r(n)$ is convex, then there exists a positive integer m and a tuple $\mathbb{A} \in \mathbb{S}_m(\mathbb{R}^g)$ such that $X \in \mathcal{D}_r(n)$ if and only if

$$I - L_{\mathbb{A}}(X) \succ 0.$$

Theorem 1.2 is a consequence of Theorem 1.5 below which asserts, under natural hypotheses, the absence of hidden singularities and the main result of [HM12]. The details are in Section 7.

Remark 1.3. In the case that \mathfrak{P}_r (or \mathcal{D}_r) is convex, it is in fact **matrix convex**. From the general theory of matrix convex sets such a set can be separated from an outlier by a linear matrix inequality [EW97] (see also [F04, F92, K, W99, WW99]). The content of Theorem 1.2 is that in the case the matrix convex set is described as the positivity set of a rational function, then a single linear matrix inequality simultaneously separates all outliers from the set.

The theory of matrix convex sets falls squarely in the realm of operator algebras and spaces [BL04, P02].

1.4. Hidden singularities. There is a stronger notion of hidden singularity which reduces to that of hidden singularity under various natural hypotheses. A hidden singularity $\chi \in \mathbb{S}_n(\mathbb{R}^g)$ of r is **well hidden** if for each m and $K \in \mathbb{S}_m(\mathbb{R}^g)$ there is a $\rho_0 > 0$ so that, for all $|\rho| < \rho_0$,

$$\chi \oplus \rho K = \begin{pmatrix} \chi & 0 \\ 0 & \rho K \end{pmatrix} \in \mathbb{S}_{n+m}(\mathbb{R}^g)$$

is also a hidden singularity.

The following is our main result on well hidden singularities. It is a stepping stone to proving the absence of hidden singularities in the presence of additional mild hypotheses as given in Theorem 1.5.

Theorem 1.4. *If r is a minimal symmetric descriptor realization (with $r(0) \succ 0$), then r has no well hidden singularities.*

Theorem 1.4 occupies a good part of this article and its proof culminates in Section 5. Proposition 7.1 gives conditions under which hidden singularities are in fact well hidden yielding the following result.

Theorem 1.5. *If r is a minimal symmetric descriptor realization with $r(0) \succ 0$, then*

- (lim) *if $\mathfrak{P}_r(n)$ is convex for each n , then $\mathfrak{P}_r(n)$ contains no hidden singularities; and*
- (alg) *the algebraic domain of r contains no hidden singularities.*

Remark 1.6. Item (alg) implicitly appears in [KvV09], though with a different proof than found here. After developing background on free rational functions in Section 6, we indicate how to obtain item (alg) from the results of [KvV09].

A very special case of Theorem 1.5 (alg) also appears in [HMOV06].

1.5. Reader's guide. The remainder of the paper is organized as follows. Background on noncommutative polynomials and minimal symmetric descriptor realizations appears in Section 2. Section 3 reminds the reader of the computation of the inverse of a block matrix via the Schur complement and describes a Fock space construction. The proof of Theorem 1.4 begins in Section 4 with the details of well hidden singularities and it concludes in Section 5. The background on free rational expressions and their algebraic domains is the subject of Section 6 and can be skipped by the reader interested only in the limit domain. Theorem 1.5 is proved at the outset of Section 7. Section 7 concludes with the proof of Theorem 1.2.

1.6. Acknowledgments. We thank Dima Kaliuzhnyi-Verbovetskyi for conversations tying this work to [KvV09]. We also thank Jaka Cimpric for providing an argument used in §6.4 and Ben Greenberg for help with the preparation of this manuscript.

2. A FEW WORDS ABOUT WORDS AND MINIMAL REALIZATIONS

Let \mathcal{W} denote the free semigroup on the g freely noncommuting formal variables $\{x_1, \dots, x_g\}$. An element w of \mathcal{W} is a **word** and takes the form

$$w = x_{i_1} \dots x_{i_k}.$$

The **empty word**, \emptyset , plays the role of the semigroup identity. There is a natural **involution** T on \mathcal{W} which reverses the order of a word,

$$w^T = x_{i_k} \dots x_{i_1}.$$

In particular, each variable itself is **symmetric** in the sense that $x_j^T = x_j$. A word w is **evaluated** at a tuple $M = (M_1, \dots, M_g)$ of $n \times n$ matrices by simply replacing x_j by M_j so that

$$w(M) = M^w = M_{i_1} \dots M_{i_k}.$$

In the case of a tuple $X = (X_1, \dots, X_g)$ of symmetric matrices, the transpose operation on words is compatible with the usual transpose operation on matrices in that $w(X)^T = w^T(X)$.

The book [BR84] is an excellent reference for additional details of the discussion of rational expressions found here. The realization of the rational expression r of Equation (1.2) is **minimal** if

$$\{(JA)^w JCh : w \in \mathcal{W}, \ h \in \mathbb{R}^\ell\}$$

spans all of \mathbb{R}^d . Equivalently, r is minimal if $u \in \mathbb{R}^d$ and

$$((JA)^w)^* JC^* u = 0$$

for all words w implies $u = 0$. (See Lemma 4.1 in [HMOV06].)

Another descriptor realization

$$r_* = D_* + C_*^T (J - L_{A_*}(x))^{-1} C_*$$

is **equivalent** to r if for each n and each X in both $\mathfrak{J}_{J_*-L_{A_*}(x)}$ and $\mathfrak{J}_{J-L_A(x)}$ we have $r_*(X) = r(X)$. Every descriptor realization is equivalent to a minimal descriptor realization. Hence, we may, and henceforth do, assume without loss of generality that the realization r is minimal. In fact, if r_* is another minimal descriptor realization equivalent to r , then, as shown in Lemma 6.2, $\text{Domlim}(r_*, n) = \text{Domlim}(r, n)$. From the definition of $\text{Domalg}(r, n)$ (which appears in Section 6) it is evident that $\text{Domalg}(r_*, n) = \text{Domalg}(r, n)$. While it is not needed in the sequel, we note that any two minimal descriptor realizations with the same D term are similar (in a precise sense we do not define here). (See [BMG05] or [HMOV06] for examples.)

3. PRELIMINARIES: SCHUR COMPLEMENTS AND A FOCK SPACE CONSTRUCTION

As preliminaries, in this section we remind the reader of the computation of the inverse of a matrix using Schur complements and introduce a Fock space construction.

3.1. Schur complements.

Lemma 3.1. *Suppose that the square matrix M has the block form,*

$$M = \begin{pmatrix} \Phi & \Omega^T \\ \Omega & \Psi \end{pmatrix},$$

Ψ is invertible and let

$$\mathcal{S} = \Phi - \Omega^T \Psi^{-1} \Omega.$$

The matrix M is invertible if and only if \mathcal{S} is invertible and moreover in this case

$$\begin{aligned} M^{-1} &= \begin{pmatrix} I & 0 \\ -\Psi^{-1}\Omega & I \end{pmatrix} \begin{pmatrix} \mathcal{S}^{-1} & 0 \\ 0 & \Psi^{-1} \end{pmatrix} \begin{pmatrix} I & -\Omega^T \Psi^{-1} \\ 0 & I \end{pmatrix} \\ (3.1) \quad &= \begin{pmatrix} \mathcal{S}^{-1} & -\mathcal{S}^{-1} \Omega^T \Psi^{-1} \\ -\Psi^{-1} \Omega \mathcal{S}^{-1} & \Psi^{-1} + \Psi^{-1} \Omega \mathcal{S}^{-1} \Omega^T \Psi^{-1} \end{pmatrix}. \end{aligned}$$

In particular, if Ψ is invertible, but \mathcal{S} is not invertible, then M is not invertible and moreover, for any vector ζ in the kernel of \mathcal{S} , the vector

$$\begin{pmatrix} \zeta \\ -\Psi^{-1}\Omega\zeta \end{pmatrix}$$

is in the kernel of M .

Remark 3.2. The matrix \mathcal{S} is the **Schur complement** of M with respect to Φ . Alternately it is the Schur complement of M **pivoting** on Ψ .

Similarly, the Schur complement of M with respect to Ψ is

$$\mathcal{S}_* = \Psi - \Omega\Phi^{-1}\Omega^T$$

and in this case the analog of Equation (3.1) is

$$\begin{aligned} (3.2) \quad M^{-1} &= \begin{pmatrix} I & -\Phi^{-1}\Omega^T \\ 0 & I \end{pmatrix} \begin{pmatrix} \Phi^{-1} & 0 \\ 0 & \mathcal{S}_*^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -\Omega\Phi^{-1} & I \end{pmatrix} \\ &= \begin{pmatrix} \Phi^{-1} + \Phi^{-1}\Omega^T\mathcal{S}_*^{-1}\Omega\Phi^{-1} & -\Phi^{-1}\Omega^T\mathcal{S}_*^{-1} \\ -\mathcal{S}_*^{-1}\Omega\Phi^{-1} & \mathcal{S}_*^{-1} \end{pmatrix}. \end{aligned}$$

Proof. Direct calculation shows that the matrix M can be written as

$$\begin{pmatrix} I & \Omega^T\Psi^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} \mathcal{S} & 0 \\ 0 & \Psi \end{pmatrix} \begin{pmatrix} I & 0 \\ \Psi^{-1}\Omega & I \end{pmatrix}.$$

The second statement is also proved by direct calculation. ■

3.2. A Fock space construction. For a given positive integer ν , let \mathcal{W}_ν denote the words of **length** at most ν . (The empty word has length 0.) Let $\mathcal{F}(\nu)$ denote the Hilbert space with orthonormal basis \mathcal{W}_ν . Thus $\mathcal{F}(\nu)$ is a truncated version of the standard Fock space on g the freely noncommuting indeterminates $\{x_1, \dots, x_g\}$. The dimension of $\mathcal{F}(\nu)$ is $\mathfrak{d} = \sum_0^\nu g^j$.

Let $S = (S_1, \dots, S_g)$ denote the shifts on $\mathcal{F}(\nu)$. Thus S_j is determined by its actions on words $w \in \mathcal{W}_\nu$ by

$$S_j w = \begin{cases} x_j w & \text{if } |w| < \nu \\ 0 & \text{if } |w| = \nu, \end{cases}$$

where $|w|$ denotes the length of the word w . It is straightforward to verify that the adjoint of S_j is determined by

$$S_j^* w = \begin{cases} \tilde{w} & \text{if } w = x_j \tilde{w} \\ 0 & \text{otherwise.} \end{cases}$$

Note that empty word, \emptyset , is a cyclic vector for the tuple S . Let $K = K(\nu)$ denote the tuple with j -th entry $K_j = S_j^* + S_j$. Thus each K_j is self adjoint and $K \in \mathbb{S}_0(\mathbb{R}^g)$.

Lemma 3.3. *Fix a word ω of length ν . If $w \in \mathcal{W}_\nu$, but $w \neq \omega$, then $K^w \emptyset$ is orthogonal to ω . In particular, given a nonzero vector ζ in a Hilbert space \mathcal{H} , there exists a mapping $Q : \mathcal{H} \rightarrow \mathcal{F}(\nu)$ such that $Q^T K^w \emptyset = 0$ if $w \neq \omega$ and $Q^T K^\omega \emptyset = \zeta$.*

Proof. Fix a word $w \neq \omega$ of length $\mu \leq \nu$ and write

$$w = x_{j_1} x_{j_2} \cdots x_{j_\mu}.$$

Because, when expanding K^w as a sum of products of the S_j and S_k^* , every term, except for S^w , contains at least one adjoint term,

$$K^w \emptyset = S^w \emptyset + m = w + m,$$

where m is a sum of words of length at most $\nu - 2$. It follows that ω is orthogonal to $K^w \emptyset$. Letting \mathcal{G} denote the span, in $\mathcal{F}(\nu)$, of the set $\{K^w \emptyset : w \in \mathcal{W}_\nu, w \neq \omega\}$ it follows that ω is orthogonal to \mathcal{G} . In particular, letting $[\omega]$ denote the span of $\{\omega\}$ and $[\omega]^\perp$ its orthogonal complement, $\mathcal{G} \subseteq [\omega]^\perp$. Define $Y : \mathcal{F}(\nu) \rightarrow H$ by $Y\omega = \zeta$ and $Y = 0$ on $[\omega]^\perp$. Then $YK^w \emptyset = 0$ for $w \neq \omega$. Choosing $Q = Y^T$ completes the proof. \blacksquare

4. WELL HIDDEN SINGULARITIES AND PERTURBATIONS

The proof of Theorem 1.4 begins here and concludes in Section 5. It proceeds by contradiction. Accordingly, suppose r , the given minimal symmetric descriptor realization as in Equation (1.2), has a well hidden singularity \mathbb{X} which is now fixed through the end of Section 5. In particular, $J - L_A(\mathbb{X})$ is singular and thus has a nontrivial kernel \mathcal{K} . Let N denote the size of \mathbb{X} so that $\mathbb{X} \in \mathbb{S}_N(\mathbb{R}^g)$.

Recall that $J - L_A(x)$ is a $d \times d$ affine linear pencil. With respect to the decomposition of $\mathbb{R}^d \otimes \mathbb{R}^N$ as $\mathcal{K} \oplus \mathcal{K}^\perp$, write

$$J - L_A(\mathbb{X}) = \begin{pmatrix} 0 & 0 \\ 0 & R \end{pmatrix}$$

and

$$-L_A(\mathbb{X}) = \begin{pmatrix} \alpha & \beta^T \\ \beta & R_1 \end{pmatrix}.$$

Letting \mathbb{V} denote the inclusion of \mathcal{K} into $\mathbb{R}^d \otimes \mathbb{R}^N$, the matrix α is given by $-\mathbb{V}^T L_A(\mathbb{X}) \mathbb{V}$. Note that R is invertible.

Lemma 4.1. *The pencil $J - L_A(\mathbb{X} + t^2 \mathbb{X})$ is invertible for t sufficiently close to 0, but $t \neq 0$.*

Proof. If not, then $\det(J - L_A(\mathbb{X} + t^2\mathbb{X}))$, being a polynomial, is identically 0. Hence $\det(J - L_A(\mathbb{X} + u\mathbb{X}))$ is identically 0 and thus 0 at $u = -1$, which gives the contradiction $\det(J) = 0$. \blacksquare

From the definitions,

$$J - L_A(\mathbb{X} + t^2\mathbb{X}) = \begin{pmatrix} t^2\alpha & t^2\beta^T \\ t^2\beta & R(t) \end{pmatrix},$$

where $R(t) = R + t^2R_1$. Since, for $t \neq 0$ but near 0, both $J - L_A(\mathbb{X} + t^2\mathbb{X})$ and $R(t)$ are invertible, Lemma 3.1 implies that

$$(4.1) \quad F(t) := [\alpha - t^2\beta^T R(t)^{-1}\beta]$$

is invertible. In particular, $\det(F(t))$ is not identically equal to zero. Thus Cramer's rule implies that there is a non-negative integer p and a nonzero matrix \mathbb{M} such that

$$(4.2) \quad \lim_{t \rightarrow 0} t^p F(t)^{-1} = \mathbb{M};$$

i.e., the limit exists and is nonzero. In fact p is an even integer, since F is really a function of t^2 . For future use, let $q = \frac{p+2}{2}$ so that $p = 2q - 2$. If α is invertible, then $p = 0$ and $q = 1$ and the arguments to come are much simpler.

4.1. A further perturbation. Let M be a positive integer and fix $K \in \mathbb{S}_M(\mathbb{R}^g)$. Let $\mathbb{S}_{M,N}(\mathbb{R}^g)$ denote the set of g -tuples $H = (H_1, \dots, H_g)$ of $M \times N$ matrices. Given such an H and real numbers s, t, ρ , let

$$(4.3) \quad \tilde{\mathbb{X}}(s, t, \rho, H) = \tilde{\mathbb{X}} = \begin{pmatrix} \mathbb{X} + t^2\mathbb{X} & st^q H^T \\ st^q H & \rho K \end{pmatrix}$$

and observe, since \mathbb{X} is a well hidden singularity, $\tilde{\mathbb{X}}$ is a hidden singularity for $t = 0$ and ρ sufficiently near 0.

Write, with respect to the decomposition of $(\mathbb{R}^d \otimes \mathbb{R}^N) \oplus (\mathbb{R}^d \otimes \mathbb{R}^M)$ as $\mathcal{K} \oplus \mathcal{K}^\perp \oplus [\mathbb{R}^d \otimes \mathbb{R}^M]$,

$$(4.4) \quad J - L_A(\tilde{\mathbb{X}}) = \begin{pmatrix} t^2\alpha & t^2\beta^T & st^q\gamma^T \\ t^2\beta & R(t) & st^q W^T \\ st^q\gamma & st^q W & J - \rho L_A(K) \end{pmatrix}.$$

Here γ and W come from decomposing $L_A(H^T)$ and so are linear maps (matrices).

Lemma 4.2. *Given H and K (as above), there is a ρ_0 such that for each $|\rho| < \rho_0$ there is an s_0 such that for each $|s| < s_0$ there is a t_0 such that for $0 < |t| < t_0$, the matrix $J - L_A(\tilde{\mathbb{X}})$ is invertible (thus $\tilde{\mathbb{X}} = \tilde{\mathbb{X}}(\rho, s, t, H, K) \in \mathfrak{J}_{J-L_A(x)}$ for such ρ, s, t).*

Proof. Since \mathbb{X} is a well hidden singularity, there exists a ρ_0 such that if $|\rho| < \rho_0$, then $\mathbb{X} \oplus \rho K$ is a hidden singularity. Moreover, ρ_0 can be chosen small enough that ρK is in the component of zero of the invertibility set of $J - L_A(x)$.

By Lemma 4.1 there is a u so that $J - L_A(\mathbb{X} + u^2\mathbb{X})$ is invertible. It follows that with ρ given and this u there is an $s_0 > 0$ such that if $|s| < s_0$ then, $J - L_A(\tilde{\mathbb{X}}(s, \rho, u, H))$ is invertible. In particular, with ρ and such an s fixed, as a function of t the matrix-valued polynomial $\tau(t) = J - L_A(\tilde{\mathbb{X}}(s, \rho, t, H))$ is invertible at $t = \pm|u|$ and hence fails to be invertible at most finitely many times. Consequently, there is a $t_0 > 0$ so that for $0 < |t| < t_0$ it is invertible (with of course ρ and then s fixed). ■

Our aim is to use the fact that for, $t = 0$ and ρ small, $\tilde{\mathbb{X}}$ is a hidden singularity and examine properties of $r(\tilde{\mathbb{X}})$ as t tends to 0 (with the other variables fixed). Accordingly, let Γ denote the lower right two by two block of the matrix in Equation (4.4). Thus

$$\Gamma = \begin{pmatrix} R(t) & st^q W^T \\ st^q W & Y \end{pmatrix},$$

where

$$(4.5) \quad Y = J - \rho L_A(K).$$

Let Δ denote the the Schur complement of Γ (relative to $R(t)$),

$$(4.6) \quad \Delta = Y - s^2 t^{2q} W R^{-1} W^T.$$

Since Y is invertible, Δ is invertible for t sufficiently close to 0.

For notational ease, write

$$(J - L_A(\tilde{\mathbb{X}})) = \begin{pmatrix} t^2 \alpha & t^2 \zeta^T \\ t^2 \zeta & \Gamma \end{pmatrix} \quad \text{with} \quad \zeta = \begin{pmatrix} \beta \\ st^{q-2} \gamma \end{pmatrix}$$

and, recalling the definition of F from Equation (4.1), let

$$(4.7) \quad F_* = F - s^2 t^{2q-2} (\gamma^T - t^2 \beta^T R^{-1} W^T) \Delta^{-1} (\gamma - t^2 W R^{-1} \beta).$$

To verify that F_* is the Schur complement for $(J - L_A(\tilde{\mathbb{X}}))$ pivoting on Γ (see Equation (3.2)), first observe that from Equation (3.1),

$$(4.8) \quad \Gamma^{-1} = \begin{pmatrix} I & -st^q R(t)^{-1} W^T \\ 0 & I \end{pmatrix} \begin{pmatrix} R(t)^{-1} & 0 \\ 0 & \Delta^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -st^q W R(t)^{-1} & I \end{pmatrix}.$$

Thus

$$(4.9) \quad \zeta^T \Gamma^{-1} \zeta = \beta^T R^{-1} \beta + (st^{q-2} \gamma^T - st^q \beta^T R^{-1} W^T) \Delta^{-1} (st^{q-2} \gamma - st^q W R^{-1} \beta).$$

Hence,

$$(4.10) \quad \alpha - t^2 \zeta \Gamma^{-1} \zeta^T = F_*.$$

Using (4.10), an application of Lemma 3.1 gives,

$$(4.11) \quad (J - L_A(\tilde{\mathbb{X}}))^{-1} = \begin{pmatrix} (t^2 F_*)^{-1} & -(t^2 F_*)^{-1} t^2 \zeta^T \Gamma^{-1} \\ -\Gamma^{-1} t^2 \zeta (t^2 F_*)^{-1} & \Gamma^{-1} + t^2 \Gamma^{-1} \zeta F_*^{-1} \zeta^T \Gamma^{-1} \end{pmatrix}.$$

Our immediate goal is to analyze $\hat{k}^T r(\tilde{\mathbb{X}}) \hat{h}$ where

$$(4.12) \quad \hat{h} = 0 \oplus \mathbf{h} \otimes h, \quad \hat{k} = 0 \oplus (\mathbf{k} \otimes k) \in [\mathbb{R}^\ell \otimes \mathbb{R}^N] \oplus (\mathbb{R}^\ell \otimes \mathbb{R}^M) = \mathbb{R}^{\ell N} \oplus (\mathbb{R}^\ell \otimes \mathbb{R}^M)$$

for given vectors $h, k \in \mathbb{R}^M$ and $\mathbf{h}, \mathbf{k} \in \mathbb{R}^\ell$. This notation we will carry throughout. From Equation (4.11),

$$(4.13) \quad \begin{aligned} \hat{k}^T [r(\tilde{\mathbb{X}}) - D \otimes I] \hat{h} &= \begin{pmatrix} 0 \\ C\mathbf{k} \otimes k \end{pmatrix}^T (J - L_A(\tilde{\mathbb{X}}))^{-1} \begin{pmatrix} 0 \\ C\mathbf{h} \otimes h \end{pmatrix} \\ &= (C\mathbf{k} \otimes k)^T [\Gamma^{-1} + t^2 \Gamma^{-1} \zeta F_*^{-1} \zeta^T \Gamma^{-1}] C\mathbf{h} \otimes h. \end{aligned}$$

4.2. Taking a limit. To analyzing the limit $\lim_{t \rightarrow 0} r(\tilde{\mathbb{X}})$, we first consider the limit $\lim_{t \rightarrow 0} t^p F_*^{-1}(t)$. For notational convenience, write

$$F_* = F - s^2 t^{2q-2} G(s, t)$$

where

$$G(s, t) = (\gamma^T - t^2 \beta^T R^{-1} W^T) \Delta^{-1} (\gamma - t^2 W R^{-1} \beta).$$

Observe, from Equation (4.6), that for $s \neq 0$ fixed,

$$G_0 = \lim_{t \rightarrow 0} G(s, t) = \gamma^T Y^{-1} \gamma,$$

which we note, in view of Equation (4.5), is independent of s . Moreover, with these notations,

$$F_* = F(t) - s^2 t^p G(s, t) = F(t) [I - s^2 t^p F(t)^{-1} G(s, t)].$$

It follows that, for $s \neq 0$ sufficiently close to 0,

$$\begin{aligned} \eta(s) &:= \lim_{t \rightarrow 0} t^p F_*^{-1} \\ &= \lim_{t \rightarrow 0} (I - (t^p F^{-1}) s^2 G)^{-1} t^p F^{-1} \\ &= \lim_{t \rightarrow 0} (I - s^2 \mathbb{M} G_0(s))^{-1} \mathbb{M}. \end{aligned}$$

Further,

$$(4.14) \quad \lim_{s \rightarrow 0} \eta(s) := \mathbb{M}.$$

Lemma 4.3. *For s fixed (and sufficiently small),*

$$(4.15) \quad \lim_{t \rightarrow 0} \begin{pmatrix} 0 & I_{dM} \end{pmatrix} \Gamma^{-1} \begin{pmatrix} 0 \\ I_{dM} \end{pmatrix} = (J - \rho L_A(K))^{-1} = Y^{-1}$$

and

$$(4.16) \quad \begin{aligned} \lim_{t \rightarrow 0} t^2 \begin{pmatrix} 0 & I_{dM} \end{pmatrix} \Gamma^{-1} \begin{pmatrix} \beta \\ st^{q-2}\gamma \end{pmatrix} F_*^{-1} \begin{pmatrix} \beta^T & st^{q-2} \end{pmatrix} \Gamma^{-1} \begin{pmatrix} 0 \\ I_{dM} \end{pmatrix} \\ = s^2 Y^{-1} \gamma \eta(s) \gamma^T Y^{-1} \\ = s^2 Y^{-1} \gamma (I - s^2 \mathbb{M} \gamma Y^{-1} \gamma^T)^{-1} \mathbb{M} \gamma^T Y^{-1}. \end{aligned}$$

Remark 4.4. Importantly, \mathbb{M} depends only upon \mathbb{X} (and the given realization for r) and not upon H, K . On the other hand, Y depends on K and γ depends upon H . Later we shall see that (4.16) is 0.

Proof. Equation (4.15) follows from Equation (4.8) and the definition of Δ given in Equation (4.6).

Moving onto Equation (4.16), note that

$$t \begin{pmatrix} 0 & I_{dM} \end{pmatrix} \Gamma^{-1} \begin{pmatrix} \beta \\ st^{q-1}\gamma \end{pmatrix} = st^{q-1} \Delta^{-1} (\gamma - t^2 W R^{-1} \beta).$$

Thus, we need to compute

$$\lim_{t \rightarrow 0} s^2 t^{2q-2} \Delta^{-1} (\gamma - t^2 W R^{-1} \beta) F_*^{-1} (\gamma^T - t^2 \beta^T R^{-1} W^T) \Delta^{-1}.$$

Now $t^{2q-2} F_*^{-1} = t^p F_*^{-1}$ converges to $\eta(s)$ as defined in Equation (4.2) and

$$\lim_{t \rightarrow 0} \Delta^{-1} (\gamma - t^2 W R^{-1} \beta) = \lim_{t \rightarrow 0} \Delta^{-1} \gamma = Y^{-1} \gamma.$$

Thus the relevant limit exists and is

$$s^2 Y^{-1} \gamma \eta(s) \gamma^T Y^{-1}$$

as claimed. ■

4.3. A limit formula. The proof of the following proposition is based upon the observation that, with \hat{h} and \hat{k} defined in (4.12),

$$\lim_{t \rightarrow 0} \hat{k}^T r(\tilde{\mathbb{X}}) \hat{h} = (\mathbf{k} \otimes k)^T r(\rho K) \mathbf{h} \otimes h$$

is independent of H (and s) for ρ sufficiently close to zero. This is because for ρ and s fixed appropriately,

$$(4.17) \quad \lim_{t \rightarrow 0} r(\tilde{\mathbb{X}}) = r(\mathbb{X} \oplus \rho K) = r(\mathbb{X}) \oplus r(\rho K).$$

Indeed this is the key use of the hypothesis that the singularity \mathbb{X} is well hidden.

Proposition 4.5. *Given $K \in \mathbb{S}_M(\mathbb{R}^g)$ and $H \in \mathbb{S}_{M,N}(\mathbb{R}^g)$, there exists a $\rho_0 > 0$ such that for each $|\rho| < \rho_0$ there is an s_0 such that for $|s| < s_0$*

$$(4.18) \quad s^2(C\mathbf{k} \otimes k)Y^{-1}\gamma(I - s^2\mathbb{M}\gamma Y^{-1}\gamma^T)^{-1}\mathbb{M}\gamma^T Y^{-1}(C\mathbf{h} \otimes h) = 0.$$

In particular,

$$(C\mathbf{k} \otimes k)Y^{-1}\gamma\mathbb{M}\gamma^T Y^{-1}(C\mathbf{h} \otimes h) = 0.$$

Proof. By (4.17) the left hand side of (4.13)

$$\hat{k}^T[r(\tilde{\mathbb{X}}) - D \otimes I]\hat{h} = (C\mathbf{k} \otimes \hat{k})^T \Gamma^{-1}(C\mathbf{h} \otimes \hat{h}) + t^2(C\mathbf{k} \otimes \hat{k})^T (\Gamma^{-1}\zeta F_*^{-1}\zeta^T \Gamma^{-1})(C\mathbf{h} \otimes \hat{h}),$$

has a limit at $t = 0$ independent of s and H . Thus the right side does too. On the right hand side the limit of the first (left most) term is handled by Equation (4.15) and is, by inspection, independent of s and H . Hence the limit of the second term on the right hand side is too. From Equation (4.16) of Lemma 4.3, the limit of this second term is the left side of Equation (4.18) which is thus independent of s . Hence the left hand side of Equation (4.18) is constantly equal to its value at 0, namely 0. Hence (4.18) holds. \blacksquare

5. PROOF OF THEOREM 1.4

Recall we have assumed that the minimal symmetric descriptor realization r has the well hidden singularity \mathbb{X} . Using the perturbation $\mathbb{X} + t\mathbb{X}$ we constructed a nonzero matrix \mathbb{M} defined by Equation (4.2). In this section we reach the contradiction $\mathbb{M} = 0$, and deduce that \mathbb{X} was not in fact a well hidden singularity thus completing the proof of Theorem 1.4.

Recall that $Y = J - \rho L_A(K)$ depends upon both ρ and K and γ depends on H . (On the other hand, \mathbb{M} depends only on \mathbb{X} .) From Proposition 4.5

$$(5.1) \quad (C\mathbf{k} \otimes k)^T Y^{-1}\gamma\mathbb{M}\gamma^T Y^{-1}(C\mathbf{h} \otimes h) = 0,$$

for all $K \in \mathbb{S}_M(\mathbb{R}^g)$, all $H \in \mathbb{S}_{M,N}(\mathbb{R}^g)$ and all ρ sufficiently small. Here, as in Proposition 4.5, $h, k \in \mathbb{R}^M$ and $\mathbf{h}, \mathbf{k} \in \mathbb{R}^\ell$.

Given a linear mapping $Q : \mathbb{R}^N \rightarrow \mathbb{R}^M$, choose $H = Q\mathbb{X} = (Q\mathbb{X}_1, \dots, Q\mathbb{X}_g)$ in Equation (5.1). Recalling the definition of \mathbb{V} given at the outset of Section 4, note that $L_A(Q\mathbb{X}) = (I_d \otimes Q)L_A(\mathbb{X})$ and further

$$\gamma = (I_d \otimes Q)L_A(\mathbb{X})\mathbb{V} = (I_d \otimes Q)(J \otimes I_N)\mathbb{V}$$

because $L_A(\mathbb{X}) = (J \otimes I_N)$ on \mathcal{K} . Henceforth we abbreviate and use Q to denote $I_d \otimes Q$ and J to denote $J \otimes I_N$ (in accordance with usual practice). Thus, by (5.1),

$$(C\mathbf{k} \otimes k)^T ([J - \rho L_A(K)]^{-1} Q J \mathbb{V} \mathbb{M} \mathbb{V}^T J Q^T [J - \rho L_A(K)]^{-1}) C\mathbf{h} \otimes h = 0.$$

It follows that

$$(5.2) \quad (C\mathbf{k} \otimes k)^T (J[I - \rho L_{AJ}(K)]^{-1} Q J V M V^T J Q^T [I - \rho L_{JA}(K)]^{-1} J) C\mathbf{h} \otimes h = 0.$$

Write Equation (5.2) as a power series in ρ and use that every coefficient must be zero to obtain, for each non-negative integer ν ,

$$(5.3) \quad \sum_{j=0}^{\nu} (JC\mathbf{k} \otimes k)^T L_{AJ}(K)^j Q J V M V^T J Q^T L_{JA}(K)^{n-j} (JC\mathbf{h} \otimes h) = 0.$$

Fix words ω_1 and ω_2 words of length ν_1 and ν_2 respectively and let $\nu = \nu_1 + \nu_2$. In the notation of Subsection 3.2, let

$$\mathcal{F} = \mathcal{F}(\nu_1) \oplus \mathcal{F}(\nu_2)$$

and let K denote the tuple of self-adjoint matrices acting on \mathcal{F} defined by

$$K_j = \begin{pmatrix} K_j(\nu_1) & 0 \\ 0 & K_j(\nu_2) \end{pmatrix}.$$

Thus, K can viewed as an element of $\mathbb{S}_M(\mathbb{R}^g)$ where M is the dimension of \mathcal{F} (and can be computed explicitly in terms of ν_1, ν_2 and g). Recall that \mathbb{X} is acting on \mathbb{R}^N and fix vectors $\zeta_1, \zeta_2 \in \mathbb{R}^N$. Using Lemma 3.3, define $Q_j : \mathbb{R}^N \mapsto \mathcal{F}(\nu_j)$ such that $Q_j^T w = 0$ if $w \neq \omega_j$ and $Q_j^T \omega_j = \zeta_j$. Define $Q : \mathbb{R}^N \rightarrow \mathcal{F}$ by

$$Q^T = \begin{pmatrix} Q_1^T & Q_2^T \end{pmatrix} : \mathcal{F} \rightarrow \mathbb{R}^N.$$

Finally let $h = \emptyset \oplus 0 \in \mathcal{F}$ and $k = 0 \oplus \emptyset \in \mathcal{F}$.

For the choices in the previous paragraph consider, for $0 \leq j \leq \nu$,

$$\begin{aligned} Q^T L_{JA}(K)^{\nu-j} (JC\mathbf{h} \otimes h) &= Q^T \sum_{|w|=\nu-j} (JA)^w JC\mathbf{h} \otimes K^w h \\ &= Q^T \sum_{|w|=\nu-j} (JA)^w JC\mathbf{h} \otimes (K^w(\nu_1)\emptyset \oplus 0) \\ &= \sum_{|w|=\nu-j} (JA)^w JC\mathbf{h} \otimes Q_1^T K^w(\nu_1)\emptyset. \end{aligned}$$

Hence,

$$Q^T L_{JA}(K)^{\nu-j} (JC\mathbf{h} \otimes h) = \begin{cases} 0 & \text{if } \nu - j < \nu_1 \\ (JA)^{\omega_1} JC\mathbf{h} \otimes \zeta_1 & \text{if } \nu - j = \nu_1 \\ * & \text{if } \nu - j > \nu_1. \end{cases}$$

Likewise,

$$Q^T L_{JA}(K)^j (JC\mathbf{k} \otimes k) = \begin{cases} 0 & \text{if } j < \nu_2 = \nu - \nu_1 \quad (\nu - j > \nu_1) \\ (JA)^{\omega_2} JC\mathbf{k} \otimes \zeta_2 & \text{if } j = \nu_2 \quad (\nu - j = \nu_1) \\ * & \text{if } j > \nu_2, \end{cases}$$

where $*$ is an expression which plays no role in the argument. In particular,

$$\begin{aligned} (JC\mathbf{k} \otimes k)^T (L_{AJ}(K))^j Q J \mathbb{V} \mathbb{M} \mathbb{V}^T J Q^T (L_{JA}(K))^{\nu-j} (JC\mathbf{h} \otimes h) \\ = \begin{cases} [(JA)^{\omega_2} JC\mathbf{k} \otimes \zeta_2]^T J \mathbb{V} \mathbb{M} \mathbb{V}^T J [(JA)^{\omega_1} JC\mathbf{h} \otimes \zeta_1] & \text{if } j = \nu_2 \\ 0 & \text{if } j \neq \nu_2. \end{cases} \end{aligned}$$

Hence, from Equation (5.3), it follows that

$$[(JA)^{\omega_2} JC\mathbf{k} \otimes \zeta_2]^T J \mathbb{V} \mathbb{M} \mathbb{V}^T J [(JA)^{\omega_1} JC\mathbf{h} \otimes \zeta_1] = 0$$

for all choices of words ω_j vectors $\zeta_j \in \mathbb{R}^N$ and $\mathbf{k}, \mathbf{h} \in \mathbb{R}^\ell$. The minimality assumption on the descriptor representation this implies that $J \mathbb{V} \mathbb{M} \mathbb{V}^T J = 0$ which in turn leads to the contradiction $\mathbb{M} = 0$ and completes the proof. (Here we have used J is invertible which gives $\mathbb{V} \mathbb{M} \mathbb{V}^T = 0$. Now \mathbb{V} is the inclusion of \mathcal{K} into $\mathbb{R}^\ell \otimes \mathbb{R}^N$, so that \mathbb{V}^T is onto \mathcal{K} .)

6. FREE RATIONAL FUNCTIONS AND THEIR DOMAINS

With the proof of Theorem 1.4 complete, it remains to establish Theorems 1.5 and 1.2. Each of these theorems naturally splits into two statements, one about the limit domain and the other about the algebraic domain of the minimal symmetric descriptor realization r . This section gives the needed background on the algebraic domain of r . Readers interested only in the limit domain can safely skip to Section 7 where the theorems are proved.

6.1. Noncommutative Polynomials. The construction of free rational expressions begins with polynomials. Recall, from Section 2 that \mathcal{W} denotes the free semigroup on the g letters $x = (x_1, \dots, x_g)$ and that T is the involution on \mathcal{W} which reverses the order of a word.

A free polynomial is then an \mathbb{R} linear combination of words from \mathcal{W} and we let $\mathbb{R}\langle x \rangle$ denote the collection of free polynomials. Hence $p \in \mathbb{R}\langle x \rangle$ has the form

$$p = \sum p_w w,$$

where the sum is finite. Evaluations on \mathcal{W} extend to $p \in \mathbb{R}\langle x \rangle$ in the obvious way as

$$p(X) = \sum p_w X^w.$$

for $X \in \mathbb{S}_n(\mathbb{R}^g)$.

A free $k_1 \times k_2$ matrix-valued polynomial p can be viewed either as a $k_1 \times k_2$ matrix with entries from $\mathbb{R}\langle x \rangle$ or as a (finite) linear combination of words with (real) $k_1 \times k_2$ matrix coefficients,

$$p = \sum P_w w.$$

In the first case one evaluates $p(X)$ entrywise and in the second case one has

$$p(X) = \sum P_w \otimes X^w.$$

Note that, for $Z_n = (0_n, 0_n, \dots, 0_n) \in \mathbb{S}_n(\mathbb{R}^g)$ where each 0_n is the $n \times n$ zero matrix, $p(Z_n) = I_n \otimes p(Z_1)$. In particular, $p(Z_n)$ is invertible for all n or no n . Because of this simple relationship, in the sequel we will often simply write $p(0)$ with the size n unspecified.

The involution on \mathcal{W} naturally extends to matrix-valued polynomials as

$$p^T = \sum P_w^T w^T.$$

The polynomial p is **symmetric** if $p^T = p$. Observe the involution is compatible with evaluation in that $p^T(X) = p(X)^T$ and, if p is symmetric, $p(X)^T = p(X)$; i.e., p takes symmetric values.

6.2. Rational Expressions and their Formal Domains. We use recursion to define the notion of a **free (noncommutative) rational expression r analytic at 0** and its value $r(0)$ at 0. This class includes free matrix-valued polynomials and $p(0)$ is the value of p at 0. If p is $k \times k$ (square) matrix-valued and the $k \times k$ matrix $p(0)$ is invertible, then p is invertible, its inverse is a free rational expression analytic at 0, and $p^{-1}(0) = p(0)^{-1}$. Formal sums and products of free rational expressions analytic at 0 with value at 0 are defined accordingly, subject to the provision that the matrix sizes are compatible. Finally, a free rational expression r analytic at 0 can be inverted provided r is $k \times k$ (square) matrix-valued and $r(0)$ is invertible. In this case, its inverse is a free rational expression, and $r^{-1}(0) = r(0)^{-1}$.

The **formal domain** in $\mathbb{S}_n(\mathbb{R}^g)$ of a free rational expression r , denoted $\text{Domfor}(r, n)$, is defined inductively. If p is a polynomial, then its formal domain is all of $\mathbb{S}_n(\mathbb{R}^g)$. If r is the inverse of the polynomial p , then the formal domain of r is $\{X \in \mathbb{S}_n(\mathbb{R}^g) : p(X) \text{ is an invertible matrix}\}$. The formal domain of a general free rational expression r is equal to the intersection of the formal domains $\text{Domfor}(r_j, n)$ for the rational expressions r_j and, as necessary, their inverses which appear in the construction of the expression r . Note that by assumption $0 \in \text{Domfor}(r, n)$. Let $\text{Domfor}(r)$ denote the sequence $(\text{Domfor}(r, n))_n$.

6.3. Equivalent Rational Expressions: Rational Functions. Note that two different expressions, such as

$$(6.1) \quad r_1 = x_1(1 - x_2x_1)^{-1} \quad \text{and} \quad r_2 = (1 - x_1x_2)^{-1}x_1$$

can be converted to each other using the rational operations described above. Thus it is natural to specify an equivalence relation on rational expressions. There are various ways of doing this and several are mentioned in a paragraph in the introduction of [KvV] and the references associated to it. The notion used here mostly is that of evaluation equivalence as found in [HMOV06] and given in more detail in [KvV09]. While it is not needed for the results here, the notion of series equivalence is used in Subsection 6.4 in connection with Remark 1.6. Next we briefly describe these notions which turn out to be the same.

It is clear how to evaluate a free rational expression r on any $X \in \text{Domfor}(r)$. We can use these evaluations to define an equivalence on free rational expressions which call evaluation equivalence. Two free rational expressions r and s analytic at 0 are **evaluation equivalent** provided $r(X) = s(X)$ for each n and each X in the Zariski open set $\text{Domfor}(r, n) \cap \text{Domfor}(s, n)$. We reiterate that the X_j are symmetric matrices.

Remark 6.1. The fact that both r and s are analytic at 0 means that for each dimension n , the 0 matrix g -tuple is in the intersection of their domains. Without this requirement that r and s are analytic at 0 it is possible that for certain n one or both of the domains $\text{Domfor}(r, n)$ or $\text{Domfor}(s, n)$ could be empty. Indeed, from the theory of polynomial identities, there are free polynomials which, for certain n , fail to be invertible on all of $\mathbb{S}_n(\mathbb{R}^g)$.

The **algebraic domain** of rational expression r (analytic at 0) in $\mathbb{S}_n(\mathbb{R}^g)$, denoted $\text{Domalg}(r, n)$, is the union of the formal domains for rational expressions s (analytic at 0) which are equivalent to r . (We note that the equivalence class of a rational expression r is what is typically called a **free rational function** \mathfrak{r} and direct the reader to [HMOV06] for further details.)

Lemma 6.2. *Let r be a rational expression analytic at 0. If $\chi \in \text{Domalg}(r, n)$, and r_* is a rational expression equivalent to r such that $\chi \in \text{Domfor}(r_*, n)$, then*

$$\lim_{X \rightarrow \chi} r(X) = r_*(\chi),$$

where the limit is taken through $X \in \text{Domfor}(r, n)$. In particular, if $\chi \in \text{Domalg}(r, n)$, then $\chi \in \text{Domlim}(r, n)$.

Proof. Self evident. ■

Another notion of equivalence comes from expanding two rational expressions r and s in a power series about 0. The rational expressions r and s are **series equivalent** if the

coefficients of their power series are the same. Given $\epsilon > 0$ and a positive integer n let

$$N_\epsilon(n) = \{X \in \mathbb{S}_n(\mathbb{R}^g) : \sum X_j^2 \prec \epsilon I_n\}.$$

The **free ϵ -neighborhood of 0** (in $(\mathbb{S}_n(\mathbb{R}^g))_n$) is the sequence of sets $(N_\epsilon(n))_n$. In [HMOV06] (see Proposition A.7) it is proved that series equivalence agrees with evaluation equivalence on some free ϵ neighborhood of 0 (inside the principal component of $\text{Domfor}(r, n) \cap \text{Domfor}(s, n)$).

6.4. The [KvV09] proof of Theorem 1.5 (alg). We conclude this section by completing the tie between this paper and [KvV09] where (alg) for rational expressions in free variables, as opposed to the symmetric variables found here, is proved.

By simply ignoring the transpose operation on polynomials and allowing evaluations on tuples of not necessarily self adjoint matrices one obtains the notion of free variables. For clarity, let $\{z_1, \dots, z_g\}$ denote freely noncommuting variables. Substituting z_j for x_j in Equation (1.2) gives,

$$(6.2) \quad r(z) = D + C^T(J - L_A(z))^{-1}C.$$

Let $M_n(\mathbb{R}^g)$ denote the set of g -tuples $Z = (Z_1, \dots, Z_g)$ of $n \times n$ matrices. The descriptor realization r is naturally evaluated at Z so long as $J \otimes I - L_A(Z)$ is invertible. In fact, this same observation holds for any rational expression giving rise to more expansive notions of domains (formal, algebraic, etc.) for rational expressions (analytic at 0) as sequences whose n -th term is a subset of $M_n(\mathbb{R}^g)$.

Conversely, if r is a rational expression in free variables which takes symmetric values when evaluated on tuples X of symmetric matrices in the component of 0 of its formal domain, then r is uniquely determined by its values on such tuples as we now explain.

What we must check to validate this claim is show that if s is another rational expression in free variables, analytic at 0, which agrees with r on symmetric matrices in the intersection of the component of 0 of their formal domains, then they also agree on any matrices where they are both defined. As noted earlier from [HMOV06]), s and r being evaluation equivalent (as rational expressions in symmetric variables) in some neighborhood of 0 implies they are series equivalent. Hence their power series about 0 in the z_j variables are identical. Thus $s(Z)$ equals $r(Z)$ whenever evaluated at any tuple Z of (not necessarily symmetric) square matrices in a free neighborhood of 0.

Let $t = r - s$. Next we show that $t(Z) = 0$ on its entire formal domain by proving this claim for every matrix dimension n separately. Let Z be a g -tuple of generic matrices of size n , that is, the k^{th} entry is a matrix with entries which are the commuting variables Z_{ij}^k . Then $t(Z) = N/\delta$ where the numerator N is a matrix polynomial and the denominator is

a scalar polynomial in the variables $\zeta := \{Z_{ij}^k : \text{all } k, i, j\}$. By assumption, δ is nonzero at some neighborhood of zero and N is zero in some neighborhood of zero. It follows that $N(\zeta)$ is identically zero.

Now that we have extended r uniquely to a free rational function \check{r} , Theorem 3.1 [KvV09] interpreted in our context implies that there are no hidden singularities in the “free” Domalg. Since our symmetric variable Domalg is contained in “free” Domalg, we conclude there are no hidden singularities in Domalg. ■

7. LMI REPRESENTATIONS

In this section we first prove Theorem 1.5 by showing that in each case the hypotheses imply that a hidden singularity is well hidden and then applying Theorem 1.4. The LMI representation results of Theorem 1.2 are then shown to be consequences of Theorem 1.5 and the main result of [HM12].

Proposition 7.1. *Suppose r is a minimal symmetric descriptor realization.*

- (lim) *If for each n the set $\mathfrak{P}_r(n)$ is convex and if $\chi \in \mathfrak{P}_r(N)$ is a hidden singularity, then χ is a well hidden singularity.*
- (alg) *If $\chi \in \text{Domalg}(r, N)$ is a hidden singularity, then χ is a well hidden singularity.*

Proof. The argument at the outset of the proof of Lemma 4.1 shows that for $t \neq 1$ sufficiently close to 1 the matrix $J - L_A(t\chi)$ is invertible.

To prove item (lim), suppose $\chi \in \mathfrak{P}_r(N)$. Thus

$$\lim_{t \rightarrow 1} r(t\chi) = r(\chi) \succ 0$$

from which it follows that $r(t\chi) \succ 0$ for t close to 1.

The assumption that $r(0) \succ 0$ implies, given $K \in \mathbb{S}_m(\mathbb{R}^g)$, there exists an $\eta > 0$ such that if $|\rho| < \eta$, then $r(\rho K) \succ 0$. It follows that, for $\delta > 0$ sufficiently small, both $(1+\delta)\chi \oplus \rho K$ and $(1-\delta)\chi \oplus \rho K$ are in $\mathfrak{P}_r(N+m)$. Hence, by the convexity assumption, so is the average, $\chi \oplus \rho K$. Hence χ is a well hidden singularity.

To prove item (alg), suppose χ is a hidden singularity of r and $\chi \in \text{Domalg}(r, N)$. In this case there exists a rational expression r_* which is equivalent to r such that χ is in the formal domain of r_* . From the definitions, the formal domain of r_* also contains a free neighborhood of 0 and hence, given $K \in \mathbb{S}_m(\mathbb{R}^g)$, there is an $\eta > 0$ such that if $|\rho| < \eta$, then $\chi \oplus \rho K$ is also in the formal domain of r_* . An application of Lemma 6.2 implies that $\chi \oplus \rho K$, is a hidden singularity of r . Thus χ is a well hidden singularity of r . ■

Corollary 7.2. *Suppose r is a minimal symmetric descriptor realization.*

- (lim) If for each n the set $\mathfrak{P}_r(n)$ is convex, then \mathfrak{P}_r contains no hidden singularities.
- (alg) The sets $\text{Domalg}(r, n)$ contain no hidden singularities.

Proof. By Theorem 1.4 any hidden singularities in the sets $\mathfrak{P}_r(n)$ and $\text{Domalg}(r, n)$ are well hidden. By Proposition 7.1 the sets $\mathfrak{P}_r(n)$ and $\text{Domalg}(r, n)$ contain no well hidden singularities. \blacksquare

Given the descriptor realization r and a positive integer n , let

$$\mathfrak{D}(r, n) = \{X \in \mathbb{S}_n(\mathbb{R}^g) : X \in \mathfrak{I}_{J-L_A(x)}(n) \text{ and } r(X) \text{ is invertible}\}.$$

For $X \in \mathfrak{D}(r, n)$ define

$$s(X) = r(X)^{-1}.$$

Thus s is the inverse of the rational expression r and the sequence $(\mathfrak{D}(r, n))_n$ is the formal domain of s . It is well known that there is a minimal descriptor realization (evaluation) equivalent to s ; i.e., there is a minimal symmetric descriptor realization

$$(7.1) \quad \tilde{r}(x) = \tilde{D} + \tilde{C}^T(\tilde{J} - L_{\tilde{A}}(x))^{-1}\tilde{C}$$

such that $\tilde{r}(X) = r(X)^{-1}$ for each n and X in the open dense set $\mathfrak{D}(r, n) \cap \mathfrak{I}_{\tilde{J}-L_{\tilde{A}}(x)}(n)$ (c.f. [HMOV06] Lemma 4.1). Recall $\mathcal{D}_r(n)$ denotes the principal component of the set $\{X \in \text{Domalg}(r, n) : r(X) \succ 0\}$ and $\mathcal{D}_r = (\mathcal{D}_r(n))_n$.

Lemma 7.3. *Suppose r is a minimal symmetric descriptor realization.*

- (lim) If each $\mathfrak{P}_r(n)$ is convex, then $\mathfrak{P}_r = \mathfrak{P}_{\tilde{r}}$. If χ is in the boundary of $\mathfrak{P}_r(n)$, then either r has a singularity at χ or \tilde{r} has a singularity at χ . In particular, $J - L_A(\chi)$ is singular or $\tilde{J} - L_{\tilde{A}}(\chi)$ is singular.
- (alg) Likewise, for each n ,

$$(7.2) \quad \{X \in \text{Domalg}(r, n) : r(X) \succ 0\} = \{X \in \text{Domalg}(\tilde{r}, n) : \tilde{r}(X) \succ 0\}.$$

In particular $\mathcal{D}_r = \mathcal{D}_{\tilde{r}}$. Further if χ is in the boundary of \mathcal{D}_r then either $J - L_A(\chi)$ or $\tilde{J} - L_{\tilde{A}}(\chi)$ is singular.

Proof. To prove item (lim) suppose $\chi \in \mathfrak{P}_r(n)$. From Corollary 7.2 we have $\chi \in \mathfrak{I}_{J-L_A(x)}(n)$. If $\chi \in \mathfrak{I}_{\tilde{J}-L_{\tilde{A}}(x)}$, then $\tilde{r}(\chi) = r(\chi)^{-1} \succ 0$ and thus $\chi \in \mathfrak{P}_{\tilde{r}}(n)$. If instead $\chi \notin \mathfrak{I}_{\tilde{J}-L_{\tilde{A}}(x)}$, then, taking the limit through $X \in \mathfrak{I}_{\tilde{J}-L_{\tilde{A}}(x)}$ and using the fact that, for X near χ , both $X \in \mathfrak{I}_{J-L_A(x)}$ and $r(X) \succ 0$ gives,

$$\lim_{X \rightarrow \chi} \tilde{r}(X) = \lim_{X \rightarrow \chi} r(X)^{-1} = r(\chi)^{-1} \succ 0.$$

It follows that χ is a hidden singularity of \tilde{r} and $\tilde{r}(\chi)$ (defined by this limit) is positive definite. Hence $\chi \in \mathfrak{P}_{\tilde{r}}(n)$ and thus $\mathfrak{P}_r(n) \subseteq \mathfrak{P}_{\tilde{r}}(n)$.

To prove the reverse inclusion, suppose $\chi \in \mathfrak{P}_{\tilde{r}}(n)$. If $\chi \in \mathfrak{I}_{J-L_A(x)} \cap \mathfrak{I}_{\tilde{J}-L_{\tilde{A}}(x)}$, then $r(\chi) = \tilde{r}(\chi)^{-1} \succ 0$ and thus $\chi \in \mathfrak{P}_r(n)$. If instead $\chi \in \mathfrak{I}_{\tilde{J}-L_{\tilde{A}}(x)}$, but $\chi \notin \mathfrak{I}_{J-L_A(x)}$, then, taking the limit through X in the (dense) set $\mathfrak{I}_{J-L_A(x)}$ and using the fact that $X \in \mathfrak{I}_{\tilde{J}-L_{\tilde{A}}(x)}$ and $\tilde{r}(X) \succ 0$ for X near χ ,

$$\lim_{X \rightarrow \chi} r(X) = \lim_{X \rightarrow \chi} \tilde{r}(X)^{-1} = \tilde{r}(\chi)^{-1} \succ 0.$$

It follows that χ is a hidden singularity of r and $r(\chi)$ (defined by this limit) is positive definite. Thus, if $\chi \in \mathfrak{I}_{\tilde{J}-L_{\tilde{A}}(x)}$, then $\chi \in \mathfrak{P}_r(n)$.

Next suppose $\chi \notin \mathfrak{I}_{\tilde{J}-L_{\tilde{A}}(x)}$ and for notational ease, let $\mathfrak{I} = \mathfrak{I}_{J-L_A(x)} \cap \mathfrak{I}_{\tilde{J}-L_{\tilde{A}}(x)}$. Note that \mathfrak{I} is open and dense. In this case,

$$\lim_{\mathfrak{I} \ni X \rightarrow \chi} \tilde{r}(X) = \tilde{r}(\chi) \succ 0.$$

Hence,

$$(7.3) \quad \lim_{\mathfrak{I} \ni X \rightarrow \chi} r(X) = \lim_{\mathfrak{I} \ni X \rightarrow \chi} \tilde{r}(X)^{-1} = \tilde{r}(\chi)^{-1} \succ 0.$$

Letting $L = \tilde{r}(\chi)^{-1}$, Equation (7.3) implies, given $\epsilon > 0$ there is a δ so that if $\|X - \chi\| < \delta$ and $X \in \mathfrak{I}$, then $\tilde{r}(X)$ is invertible and $\|r(X) - L\| < \epsilon$. Now suppose only that $X \in \mathfrak{I}_{J-L_A(x)}$ and $\|X - \chi\| < \delta$. In this case, using continuity of r at X and the fact that \mathfrak{I} is open and dense, there is an $Y \in \mathfrak{I}$ such that $\|Y - \chi\| < \delta$ and $\|r(X) - r(Y)\| < \epsilon$. Hence,

$$\|r(X) - L\| < 2\epsilon.$$

It follows that

$$\lim_{\mathfrak{I}_{J-L_A(x)} \ni X \rightarrow \chi} r(X) = L.$$

As also $L \succ 0$, it follows that $\chi \in \mathfrak{P}_r(n)$ and thus $\mathfrak{P}_{\tilde{r}}(n) \subseteq \mathfrak{P}_r(n)$.

To complete the proof of item (lim), suppose χ is in the boundary of \mathfrak{P}_r . In this case, either

$$\lim_{X \rightarrow \chi} r(X)$$

exists and is both positive semidefinite and singular, or the limit fails to exist (and necessarily $X \notin \mathfrak{I}_{J-L_A(x)}$). In this second case, r has a singularity at χ . In the first case, \tilde{r} must have a singularity at χ . In particular, either $J - L_A(\chi)$ is singular, or $\tilde{J} - L_{\tilde{A}}(\chi)$ is singular.

Now we turn to the proof of item (alg). First suppose $X \in \text{Domalg}(r, n)$ and $r(X) \succ 0$. By Corollary 7.2 we see X is in the formal domain of r . It follows that the rational expression r^{-1} is defined at X and hence, by Corollary 7.2 (alg), $X \in \text{Domfor}(\tilde{r}, n)$. Moreover, $\tilde{r}(X) \succ 0$ since $r^{-1}(X) \succ 0$. Hence, $\mathcal{D}_r \subseteq \mathcal{D}_{\tilde{r}}$.

To establish the reverse inclusion in Equation (7.2), fix $X \in \mathcal{D}_{\tilde{r}}(n)$. Let \hat{r} denote the rational expression $(\tilde{r})^{-1}$. In particular, \hat{r} is equivalent to r . By Corollary 7.2 applied to

\tilde{r} , the tuple X is in fact in the formal domain of \tilde{r} . Since also $X \in \mathcal{D}_{\tilde{r}}(n)$ it follows that $\tilde{r}(X) \succ 0$ (and is thus invertible). Thus X is in the formal domain of \hat{r} . Hence X is in the algebraic domain of r and consequently, by Corollary 7.2, X is in the formal domain of r (equivalently $J - L_A(X)$ is invertible). Further, $r(X) = \hat{r}(X) = \tilde{r}(X)^{-1} \succ 0$ and the reverse inclusion, $\mathcal{D}_{\tilde{r}} \subseteq \mathcal{D}_r$, in Equation (7.2) is established.

To complete the proof of item (alg), suppose χ is in the boundary of $\mathcal{D}_r(n)$. If $\chi \notin \text{Domalg}(r, n)$, then $J - L_A(\chi)$ is not invertible. Thus, it may be assumed that $\chi \in \text{Domalg}(r, n)$, but $r(\chi)$ is both positive semidefinite and singular. If $\chi \in \text{Domalg}(\tilde{r}, n)$, then

$$\lim_{X \rightarrow \chi} r(X)^{-1} = \lim_{X \rightarrow \chi} \tilde{r}(X),$$

where the limit is taken through $X \in \mathcal{I}_r$, giving the contradiction that $r(\chi)$ is invertible. Hence $\tilde{J} - L_{\tilde{A}}(\chi)$ fails to be invertible. \blacksquare

Proof of Theorem 1.2. To prove item (lim), suppose \mathfrak{P}_r is convex.

Let \mathcal{I}_P° denote the component of 0 of the invertibility set of the affine linear pencil

$$P(x) = J \oplus \tilde{J} - L_{A \oplus \tilde{A}}(x) = [J - L_A(x)] \oplus [\tilde{J} - L_{\tilde{A}}(x)].$$

Let $X \in \mathbb{S}_n(\mathbb{R}^g)$ be given. If $X \in \mathfrak{P}_r$, then for each $0 \leq t \leq 1$ we have $tX \in \mathfrak{P}_r$ by convexity. By Corollary 7.2 $J - L_A(tX)$ is invertible for each t , since \mathfrak{P}_r contains no hidden singularities of r . By Lemma 7.3, $\mathfrak{P}_r = \mathfrak{P}_{\tilde{r}}$ and again an application Corollary 7.2 implies that $\tilde{J} - L_{\tilde{A}}(tX)$ is invertible for $0 \leq t \leq 1$. It follows that X is in the component of zero of the invertibility set of \mathcal{I}_P and hence $\mathfrak{P}_r \subseteq \mathcal{I}_P^\circ$.

Now suppose X is in the component of zero of the invertibility set of P . Choose any path $F(t)$ connecting zero to X which lies entirely in the component of zero of \mathcal{I}_P . If this path does not lie entirely in $\mathfrak{P}_r = \mathfrak{P}_{\tilde{r}}$, then there is a first point t_0 such that $F(t_0)$ is in the boundary of \mathfrak{P}_r . Hence, by Lemma 7.3, $P(F(t_0))$ must be singular. But then $F(t_0)$ is not in the invertibility set of P and the path F does not lie entirely in the component of zero of the invertibility set of P , a contradiction. Hence $X \in \mathfrak{P}_r$. Thus $\mathcal{I}_P^\circ \subseteq \mathfrak{P}_r$ and the equality $\mathcal{I}_P^\circ = \mathfrak{P}_r$ is proved.

We have established \mathfrak{P}_r is the component of zero of the invertibility set of the symmetric matrix valued polynomial P . Assuming also that \mathfrak{P}_r is bounded, the main result of [HM12] says that \mathcal{I}_P° has an LMI representation; i.e., there exists \mathbb{A} such that X is in the component of zero of the set where P is invertible if and only if $I - L_{\mathbb{A}}(X) \succ 0$.

The proofs of items (alg) and (lim) are similar as we see now. Suppose \mathcal{D}_r is convex. If $X \in \mathcal{D}_r$, then for each $0 \leq t \leq 1$ we have $tX \in \mathcal{D}_r$ by convexity. By Corollary 7.2 the matrix $J - L_A(tX)$ is invertible for each t , since \mathcal{D}_r contains no hidden singularities of r . By Lemma 7.3(alg), $\mathcal{D}_r = \mathcal{D}_{\tilde{r}}$ and again an application of Corollary 7.2 implies that $\tilde{J} - L_{\tilde{A}}(tX)$

is invertible for $0 \leq t \leq 1$. It follows that X is in the component of zero of the invertibility set of the pencil P . Hence $\mathcal{D}_r \subseteq \mathfrak{I}_P^\circ$.

Now suppose X is in the component of zero of the invertibility set of P . Choose any path $F(t)$ connecting 0 to X which lies entirely in the component of zero of P . If this path does not lie entirely in $\mathcal{D}_r = \mathcal{D}_{\tilde{r}}$, then there is a first point t_0 such that $F(t_0)$ is in the boundary of \mathcal{D}_r . Hence, by Lemma 7.3 (alg), $P(F(t_0))$ must be singular. But then $F(t_0)$ is not in the invertibility set of P and the path F does not lie entirely in the component of zero of the invertibility set of P , a contradiction. Hence $X \in \mathcal{D}_r$. Thus $\mathfrak{I}_P^\circ \subseteq \mathcal{D}_r$ and the equality $\mathfrak{I}_P^\circ = \mathcal{D}_r$ is proved.

We have established that \mathcal{D}_r is the component of zero of the invertibility set of the symmetric matrix valued polynomial P . Assuming also that \mathcal{D}_r is bounded, the main result of [HM12] says that \mathfrak{I}_P° has an LMI representation; i.e., there exists \mathbb{A} such that X is in the component of zero of the set where P is invertible if and only if $I - L_{\mathbb{A}}(X) \succ 0$. ■

REFERENCES

- [BMG05] J.A. Ball, T. Malakorn, G. Groenewald, *Structured noncommutative multidimensional linear systems*, SIAM J. Control Optim. 44 (4) (2005) 14741528. 7
- [BR84] J. Berstel, and C. Reutenauer, *Rational Series and their Languages*, EATCS Monographs on Theoretical Computer Science, Springer, 1984 6
- [BL04] Blecher, David P.; LeMerdy, Christian, *Operator Algebras and Their Modules An operator space approach*, London Mathematical Society Monographs 30, 2004. 5
- [EW97] Effros, Edward G.; Winkler, Soren, *Matrix convexity: operator analogues of the bipolar and Hahn-Banach theorems*, J. Funct. Anal. 144 (1997), no. 1, 117152. 5
- [F04] Farenick, Douglas R., *Pure matrix states on operator systems*, Linear Algebra Appl. 393 (2004), 149173. 5
- [F92] Farenick, Douglas R., *C-convexity and matricial ranges*, Canad. J. Math. 44 (1992), no. 2, 280297. 5
- [HMV06] J. W. Helton, S. A. McCullough and V. Vinnikov, *Noncommutative Convexity Arises from Linear Matrix Inequalities*, J. Funct. Anal. 240 (2006), no. 1, 105-191. 6, 7, 18, 19, 21
- [HM12] J. W. Helton and S. A. McCullough, *Every Free Convex Basic Semi-Algebraic Set has an LMI Representation*, Annals of Math, to appear. 1, 2, 5, 20, 23, 24
- [KvV09] Kaliuzhnyi-Verbovetskyi, Dmitry; and Vinnikov, Victor, *The singularities of rational functions and factorization: The commutative and noncommutative settings*, Linear Algebra and its Appl. 439 (2009), 869–889. 6, 18, 19, 20
- [K] Kleski, Greg, *Boundary representations and pure completely positive maps*, J. Operator Theory, to appear. 5
- [P02] Paulsen, Vern I., *Completely Bounded Maps and Operator Algebras*, Cambridge University Press, 2002. 5
- [W99] Winkler, Soren, *The non-commutative Legendre-Fenchel transform*, Math. Scand. 85 (1999), no. 1, 3048. 5

- [WW99] Webster, Corran; Winkler, Soren, *The Krein-Milman theorem in operator convexity*, Trans. Amer. Math. Soc. 351 (1999), no. 1, 307322. 5

J. WILLIAM HELTON, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SAN DIEGO
E-mail address: `helton@math.ucsd.edu`

SCOTT McCULLOUGH, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, GAINESVILLE
E-mail address: `sam@math.ufl.edu`